

# ON MODEL THEORY, ZILBER AND PHYSICS

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## Abstract

We study B. Zilber's method of calculating the Feynman propagator for the free particle and harmonic oscillator. Our interpretation of the method is to build a model (essentially by taking a metric ultraproduct) where we actually have eigenvectors for both the position and momentum operators.

In Jouko Väänänen's 60th birthday meeting B. Zilber gave a talk on the use of model theory in quantum physics. We got interested and tried to read [Zi] in which Zilber calculates the Feynman propagator for the free particle and the harmonic oscillator. But we failed to follow the calculations. However, the paper suggests various ways of constructing models that resemble the Hilbert space operator model used in quantum physics for a single particle (this kind of hyperfinite approach is not completely new, see e.g. [Ya]). In addition, we noticed that by using number theory, in the case of the free particle, the Feynman propagator is easy to calculate in these models (the case of the harmonic oscillator is harder). We did several straight forward such constructions and each time the model calculated the propagator incorrectly. So we asked, is there a way of constructing a model that calculates the propagator correctly and how would the physics look like in such a model? In the first section of this paper we will give one such construction. In the second section we use this model to calculate the Feynman propagator for the free particle. In the third we discuss some of the properties of the model and the implications these have on the (physical) space it describes. In the fourth section we study the harmonic oscillator and make some further remarks about similarities between our model and the one used in quantum physics.

In quantum mechanics physical observables are described by self adjoint operators on an infinite-dimensional complex Hilbert space. The unit sphere of the space corresponds to the possible states of the system and the possible outcomes of measurements are the eigenvalues of the operators. If a self adjoint operator has non-degenerate eigenvalues, the corresponding eigenvectors will be orthogonal with real eigenvalues. Physicists work under the assumption that there not only are eigenvalues, but that the eigenvectors of any of the operators considered span the whole space. This is called 'inserting a complete set of states'. So they work in a Hilbert space spanned by the eigenvectors corresponding to the possible outcomes of measurements. Linear combinations of these are considered 'superpositions', states in

which the observable is not yet determined until it is measured (upon which the state 'collapses' onto one of the eigenstates). The coefficients in the linear combinations determine the probability that the state will collapse onto the corresponding eigenstate when the observable is measured.

The assumption of a complete set of states clashes with the convention of working in a separable Hilbert space. In fact, it is easy to see that e.g. the standard position operator (see below) does not have any eigenvectors. Mathematical physicists override this problem by not assuming the existence of eigenvectors. Instead they study regions of the spectrum within which the measurements can fall. The motivation for this is that one in reality cannot measure point values anyway, but e.g. only whether a particle is inside a given area (the detector) of space. Thus notions involving eigenvectors are replaced with methods based on the spectral theorem (for unbounded self adjoint operators). Let us look at this closer in the case one wants to calculate the Feynman propagator for the free particle.

The method popular among physicists is explained in [Zi]. In [Lu] this is done differently. First of all J. Lukkarinen does not calculate the propagator as defined below, instead he calculates the kernel of the time evolution operator. Mathematically these are different things but they are used the same way when one wants to calculate the movement of the particle and thus they should have the same value (otherwise the two approaches give different physics). The first thing to do is to find the time evolution operator  $K^t$  (i.e. an operator that describes the evolution of the system in such a way that the Schrödinger equation holds). In the approach described in [Zi], spectral theory is used but Lukkarinen relies more on Fourier analysis. Below in the spaces  $H_N$  we find the time evolution operator using the spectral theory method, only now the needed spectral theory is trivial since the dimensions of the spaces are finite.

Then in the [Zi] approach an integral value for the propagator is calculated using Feynman path integral and by a straight forward calculation Lukkarinen shows that for Schwartz test functions  $f$ ,

$$K^t(f)(x) = \int_{\mathbf{R}} dk e^{i2\pi k(x-y) - it\frac{1}{2}(2\pi k)^2}$$

(here Lukkarinen has done some scaling and the units are chosen so that both  $\hbar$  and the mass of the particle are one). Finally, in [Zi] a value for the integral is found by a leap of faith but Lukkarinen argues as follows: Let us denote by  $K_*^t$  the operator defined using the correct answer as the kernel. So one needs to show that  $K^t = K_*^t$ . For this for all positive reals  $\epsilon$  one defines operators

$$K_\epsilon^t(f)(x) = \int_{\mathbf{R}} dk e^{i2\pi k(x-y) - (it+\epsilon)\frac{1}{2}(2\pi k)^2}$$

(this is called regularizing  $K^t$ ). By using dominant convergence theorem, Fubini and the known values for Gauss integrals, for Schwartz test functions  $f$  it is possible to show that

$$K^t(f) = \lim_{\epsilon \rightarrow 0} K_\epsilon^t(f) = K_*^t(f).$$

This suffices.

Although Zilber aims further, the work in [Zi], can be seen as a study of possibilities of finding a model in which some of the features of the calculations that physicists do, make not only physical sense but also mathematical sense: build a model in which we indeed do have the eigenvectors that can be used to do calculations. At least in what we do below, the cost is two fold: Of course, the model we construct can not fall into the scope of Stone-von Neumann theorem (see below). What fails is that the representation  $t \mapsto V^t$  is not continuous in the model, in fact, it is very turbulent. Also, although the model calculates the propagator correctly i.e. the movement of the particle in the model is what it should be, in the bigger picture the physics in the model is, say, unusual.

In the cases of which we are interested, in the standard complex  $L_2$  space (of square integrable functions from  $\mathbf{R}$  to  $\mathbf{C}$ ) the operators are the position operator  $Q(f)(x) = xf(x)$  and the momentum operator  $P(f)(x) = -i\hbar(df/dx)(x)$  (here  $\hbar = h/2\pi$ , where  $h$  is the Planck constant and we restrict to the case where the space in which the particle lives is of dimension one). However, in no finite dimensional Hilbert space one can find self adjoint operators  $Q$  and  $P$  satisfying the commutation relation  $[Q, P] = i\hbar$ . Thus instead of studying  $Q$  and  $P$  directly we study their Fourier transforms  $U^t = e^{itQ}$  and  $V^t = e^{itP}$ ,  $t \in \mathbf{R}$ .

This Hilbert space operator model from quantum physics for a single particle in one dimensional space (from the point of view of e.g. the Feynman propagator the dimension assumption is w.l.o.g.) is governed by two theorems. The first one is Stone's theorem which states that if for all  $t \in \mathbf{R}$ ,  $U^t$  is a unitary operator in a (complex) Hilbert space  $H$ ,  $t \mapsto U^t$  is continuous (i.e.  $t \mapsto U^t(x)$  is continuous for all  $x \in H$ ) and  $U^{t+t'} = U^t U^{t'}$ , then there is a unique self adjoint operator  $Q$  such that  $U^t = e^{itQ}$  for all  $t \in \mathbf{R}$  (typically the self adjoint operator is not bounded and so by  $e^{itQ}$  we mean the continuation to the whole space). We will call the maps  $t \mapsto U^t$  as above continuous unitary representations of  $(\mathbf{R}, +)$ . The other theorem is the Stone-von Neumann theorem. It states that the class of Hilbert spaces  $H$  with continuous unitary representations  $t \mapsto U^t$  and  $t \mapsto V^t$  of  $(\mathbf{R}, +)$  with the property  $V^w U^t = e^{ihtw} U^t V^w$  is essentially categorical (in the sense of model theory i.e. 'unitarily') in every cardinality. However, we want to point out that if  $t \mapsto U^t$  and  $t \mapsto V^t$  are as above and  $a$  is e.g. a positive real then so are  $t \mapsto U_*^t$  and  $t \mapsto V_*^t$  where  $U_*^t = U^{at}$  and  $V_*^t = V^{a^{-1}t}$  and that this kind of scaling may have an effect in calculations. We want to point out also that although, following Zilber, we use  $t$  as the parameter in these representations, we do not think of it as time, it is just a parameter. But we also use  $t$  to denote the time, it should be clear from the context which we mean.

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## 1. The model

Now following roughly Zilber we start to build a model. Since there is room for

scaling, we pick two positive reals  $a$  and  $b$  for the scaling and in the end we will study the question of the effects of this scaling. Also in our model the space (in which the particle lives - not the Hilbert space) will be of finite length. There does not seem to be a way of avoiding this in this construction (if one wants a model that calculates the Feynman propagator correctly). In the end we will study the size of the space.

For all natural numbers  $N > 1$ , let  $H_N$  be a vector space over the complex numbers  $\mathbf{C}$  with basis  $\{u(x) \mid x \in \mathbb{N}, x < N\}$ . In the end, we will take an ultraproduct (and a metric ultraproduct) of these and so when we claim that something is true of  $H_N$ , we mean that it is true assuming  $N$  is a sufficiently divisible natural number. Similarly, when we define something, it is enough if it makes sense in those  $H_N$  in which  $N$  is a sufficiently divisible natural number.

We make Hilbert spaces out of the vector spaces  $H_N$  by defining an inner product  $\langle \cdot \mid \cdot \rangle$  so that  $\langle u(x) \mid u(x') \rangle = 0$  if  $x \neq x'$  and otherwise  $\langle u(x) \mid u(x') \rangle = 1$  (and for complex numbers  $c$  and  $d$ ,  $\langle cu(x) \mid du(x) \rangle = \bar{c}d$ ).

By  $q = q_N$  we denote the complex number  $e^{i2\pi/N}$  and denote

$$v(x) = (1/N)^{1/2} \sum_{y=0}^{N-1} q^{xy} u(y).$$

Notice that for  $x \neq x'$ ,

$$\langle v(x) \mid v(x') \rangle = (1/N) \sum_{y=0}^{N-1} e^{i2\pi zy/N}$$

for some integer  $z \neq 0$  with  $|z| < N$  and thus  $\langle v(x) \mid v(x') \rangle = 0$ . Similarly  $\langle v(x) \mid v(x) \rangle = 1$  and so  $\{v(x) \mid x \in \mathbb{N}, x < N\}$  is another orthonormal basis of  $H_N$ . Furthermore, similarly, one can see that

$$u(x) = (1/N)^{1/2} \sum_{y=0}^{N-1} q^{-xy} v(y)$$

since

$$u(x) \mapsto (1/N)^{1/2} \sum_{y=0}^{N-1} q^{-xy} v(y)$$

and

$$v(x) \mapsto (1/N)^{1/2} \sum_{y=0}^{N-1} q^{xy} u(y)$$

are inverses of each other.

For all real numbers  $t$ , let  $t^u = at/2\pi$  and  $t^v = bt/2\pi$ . Also for all real numbers  $t$ , we define operators  $U^t$  and  $V^t$  as follows: For all  $0 \leq x < N$ ,  $U^t(u(x)) = q^{xt^u} u(x)$  and  $V^t(v(x)) = q^{xt^v} v(x)$ . Notice that if  $t^v$  is a natural number, then

$$(1) V^t(u(x)) = u(x - t^v)$$

where the sum is taken 'modulo  $N$ ' i.e.  $-y = N - y$  if  $N \geq y > 0$  (since

$$u(x - t^v) = (1/N)^{1/2} \sum_{y=0}^{N-1} q^{-(x-t^v)y} v(y)$$

and

$$V^t(u(x)) = V^t((1/N)^{1/2} \sum_{y=0}^{N-1} q^{-xy} v(y)) = (1/N)^{1/2} \sum_{y=0}^{N-1} q^{-xy} q^{t^v y} v(y).$$

Let us recall what we have:  $U^t$  and  $V^w$  are unitary onto operators,  $u(x)$  is an eigenvector of  $U^t$  with eigenvalue  $e^{ixat/N}$  and  $v(x)$  is an eigenvector of  $V^w$  with eigenvalue  $e^{ixbw/N}$  and assuming that both  $t^u$  and  $w^v$  are natural numbers,  $V^w U^t(u(x)) = q^{xt^u} u(x - w^v)$  and  $U^t V^w(u(x)) = q^{(x-w^v)t^u} u(x - w^v)$  i.e.  $V^w U^t = q^{w^v t^u} U^t V^w = e^{iabtw/2\pi N} U^t V^w$ .

Now it is time to look at the commutator law. We want that

$$V^w U^t = e^{i\hbar tw} U^t V^w$$

(Zilber uses here  $V^w U^t = e^{2\pi i \hbar tw} U^t V^w$ .) So should we require that  $ab = Nh$ ? We could do this (making  $a$  or  $b$  or both depend on  $N$ ) and to a point things would go much as how we actually do. However we follow, in a sense, Zilber i.e. we require that  $ab = h$ ,  $a$  and  $b$  do not depend on  $N$  and consider  $h_N = h/N$  in the model  $H_N$ . How is this possible? We scale units, i.e. units in  $H_N$  are not the same as in the 'real world'. So what unit(s) do we scale? We scale time or mass or both, it will turn out that this choice makes no difference. This is because everywhere in our calculations these units cancel out i.e. after introducing  $V_*^t$  we can forget this scaling (e.g. we let 1 unit of time be  $N$  units of time in  $H_N$ , keep in mind that the unit of  $h$  is that of  $mass \times length^2/time$ ). This is not the case if we scale length. In fact this would have small unwanted consequences beyond the trouble of keeping this scaling in mind all the time.

As we consider  $V^t = e^{itP}$ , where  $P$  is the momentum operator, and the eigenvalue of  $V^t$  at  $v(x)$  is  $e^{ibtx/N}$ , we consider  $v(x)$  an eigenvector of  $P$  with eigenvalue  $bx/N$  which is interpreted as momentum. Similarly, we consider  $U^t = e^{itQ}$ , where  $Q$  is the position operator (with eigenvalues  $ax/N$ ). Now keeping in mind the scaling, in the units of the real world, the  $P$ -eigenvalue of  $v(x)$  is  $bx$  (the unit of momentum is that of  $mass \times length/time$ ), however, the  $Q$ -eigenvalues are not scaled (because we do not scale length). Thus we also look at the operators that directly give the eigenvalue in the right units, i.e. the operators  $V_*^t = (V^t)^N$  for which  $V_*^t(v(x)) = e^{ibtx} v(x)$ . And then if both  $t^u$  and  $Nw^v$  are natural numbers, then  $V_*^w U^t = e^{iabtw/2\pi} U^t V_*^w = e^{i\hbar tw} U^t V_*^w$ .

**Remark.** Of course, as with  $a$  and  $b$ , we could also do an infinite scaling here (other than the one we chose above): We could choose for each  $N$  positive integers  $N_a$  and  $N_b$  such that  $N_a N_b = N$  and work with operators  $U_{**}^t = (U^t)^{N_a}$  and  $V_{**}^t = (V^t)^{N_b}$ . If in addition we choose these integers and the ultrafilter (see below) so that for all  $n$  the set of those  $N$  such that  $n$  divides both  $N_a$  and  $N_b$  belongs to the filter, then the commutator law holds in the ultraproduct for all  $t$  and  $w$  such that both  $t^u$  and  $w^v$  are rational numbers and the space in which the particle moves becomes infinite (see below). However we do not do this because in such a model the propagator for a free particle gets an infinitesimal value (and is zero in the metric ultraproduct).

Now we take an ultraproduct of these structures. There are two kinds of ultraproducts we may consider. First the ordinary one with equivalence classes defined by the ultrafilter. The second is the metric ultraproduct, where one first takes the product, then defines norms as ultralimits of the coordinatewise norms, then consider only the elements with finite norms and finally mods out the infinitesimals.

We would like to take a metric ultraproduct but unfortunately it does not work. First of all, the function  $(t, x) \mapsto V_*^t(x)$  does not have a modulus of uniform continuity. In fact it is easy to see that infinitesimal changes in the argument may result in a non-infinitesimal change in the value. So the function is far too turbulent. (For fixed  $t$ ,  $V^t$  is unitary and thus it indeed has a modulus of uniform continuity). This same problem arises when we introduce the time evolution operator. And if we just forget the operators  $V_*^t$  and take the metric ultraproduct we lose the commutator law information, the operators  $U^t$  and  $V^t$  commute in the metric ultraproduct (but not in the usual ultraproduct where the unit of time is infinitesimal). So what we do is that we take the usual ultraproduct and then inside that we find the metric ultraproduct and just live with the fact that some of our functions are well-defined only in the usual ultraproduct.

Let  $X = \{x \in \mathbb{N} \mid x > 1\}$  and  $D$  be an ultrafilter on  $X$  such that for all  $n > 0$ ,  $X_n \in D$ , where the set  $X_n$  consists of those  $N \in X$  which are divisible by  $n$ . We think of our structures  $H_N$  as two-sorted structures: in one sort we have the elements of the Hilbert space and in the other one the complex numbers  $\mathbf{C}$ . The vocabulary consists of the addition in the Hilbert space, the inner product from  $(H_N)^2$  to  $\mathbf{C}$ , the scalar multiplication  $\mathbf{C} \times H_N \rightarrow H_N$ , the field structure on  $\mathbf{C}$ , the norm  $|\cdot|$  in  $\mathbf{C}$  (or the complex conjugation), the complex exponentiation and constants for  $i$ ,  $h$  and  $\pi$  and, of course, the operators  $U^t, V^t$  and  $V_*^t$ ,  $t \in \mathbf{R}$ . When we introduce the time evolution operators, we add them to this list. Then we let  $(H_*, \mathbf{C}_*)$  be the ultraproduct  $\prod_{N \in X} (H_N, \mathbf{C}) / D$ . Notice that  $\mathbf{C}_*$  with the field structure is isomorphic to the field of complex numbers but  $\mathbf{R}_*$  i.e. the image of the norm closed under subtraction is not isomorphic to the reals. It is a real closed subfield of  $\mathbf{C}_*$  but contains e.g. infinitesimals. However  $\mathbf{C}_* = \mathbf{R}_*[i]$ .  $H_*$  looks a lot like a Hilbert space: it is a vector space over  $\mathbf{C}_*$  with something that looks like an inner product, i.e. the inner product satisfies everything one requires from an inner product if as complex conjugation one uses the one that arises from the fact that

$\mathbf{C}_* = \mathbf{R}_*[i]$ . From this we get a 'norm' on  $H_*$  the usual way, which now can get infinitesimal values. When we talk about the norm of an element of  $H_*$ , it is this that we mean. Similarly, the norm of elements of  $\mathbf{C}_*$  is the one that is in our vocabulary which is the same as the one one gets from our complex conjugation. Notice that now the completeness of  $H_N$  loses its meaning.

Next we find the metric ultraproduct. We let  $\mathbf{C}'_*$  be the set of those elements  $q \in \mathbf{C}_*$  whose norm is less than some natural number  $n$  ( $= 1 + 1 + \dots + 1$ ).  $H'_*$  is defined similarly. Notice that  $H'_*$  and  $\mathbf{C}'_*$  are closed under all functions of our vocabulary. Then we define an equivalence relation  $E$  so that for  $q, q' \in \mathbf{C}'_*$ ,  $qEq'$  if  $|q - q'| < 1/n$  for all natural numbers  $n$  and on  $H'_*$ ,  $E$  is defined similarly. We denote  $H = H'_*/E$  and  $\mathbf{C} = \mathbf{C}'_*/E$  (we will justify this notation soon). As mentioned above, excluding  $(t, x) \mapsto V_*^t(x)$ , we can define all functions from our vocabulary by using representatives from the equivalence classes (e.g.  $U^t(x/E) = U^t(x)/E$ ). Let  $j$  be the canonical embedding of complex numbers to  $\mathbf{C}_*$  and then it is easy to see that  $\mathbf{C}$  is the set of all  $j(q)/E$ , where  $q$  is a complex number (bounded closed subsets of complex numbers are compact) i.e.  $q \mapsto j(q)/E$  is an isomorphism from the real complex numbers to (our)  $\mathbf{C}$  (in the vocabulary that consist of all the structure we have put on  $\mathbf{C}$  alone). Thus we think of our  $\mathbf{C}$  as the complex numbers. In the context of  $\mathbf{C}_*$ , by a complex number  $q$  we mean  $j(q)$ . And now  $H$  is a complex Hilbert space. From now on we work in  $H$  keeping in mind that  $(t, x) \mapsto V_*^t(x)$  and later  $(t, x) \mapsto K^t(x)$  are well-defined only in  $H_*$  i.e. we move to  $H_*$  if necessary. It will be clear from the context in which structure we work.

Now the function  $t \mapsto U^t$  is a continuous unitary representation of  $(\mathbf{R}, +)$  and locally also  $t \mapsto V_*^t$  is (see Section 4) and for all  $t$  and  $w$  such that  $t^u$  is an integer and  $w^v$  is a rational number,  $V_*^w U^t = e^{ihtw} U^t V_*^w$  (assuming  $ab = h$ ). Also assuming  $a^{-1}x$  is a rational number  $< 1$ , (e.g.)  $u(x) = (u(a^{-1}Nx) | N \in X)/D/E$  is an eigenvector of each  $U^t$  with eigenvalue  $e^{itx}$ . Similarly, (e.g.)  $v(x) = (v(x) | N \in X)/D$  is an eigenvector of each  $V_*^t$  with eigenvalue  $e^{itbx}$ . As pointed out in Remark above, the picture would look a lot better had we used also infinite scaling.

## 2. The free particle

So let us consider the time evolution operator for the free particle. For a time independent Hamiltonian  $H$  it is

$$K^t = e^{-itH/\hbar}.$$

We wish to calculate the Feynman propagator  $\langle x_1 | K^t | x_0 \rangle$  for the free particle in our model (where  $\langle x_1 | K^t | x_0 \rangle$  is the inner product between  $|x_1\rangle$  i.e.  $Q$ -eigenvector with eigenvalue  $x_1$  and  $K^t | x_0 \rangle = K^t(|x_0\rangle)$ ).

For this we need to define the time evolution operators  $K^t$  in the spaces  $H_N$ : We let

$$K^t(v(x)) = e^{-it(bx)^2/2\hbar m} v(x),$$

for all  $0 \leq x < N$  (keep in mind that  $bx$  is the  $P$ -eigenvalue of  $v(x)$  and as Hamiltonian we use  $P^2/2m$ , where  $m$  is the mass of the particle, see [Ze]; in [Zi] there is probably a typo here). The right answer according to physicists is

$$(m/2\pi i\hbar t)^{1/2} e^{im(x_0-x_1)^2/2\hbar t}$$

but in our calculations the answer depends on our scaling and there is only one choice that gives the right answer, namely that  $a = \hbar t/m$  and  $b = m/t$  (keep in mind that  $ab$  should be  $\hbar$ ) and we do the calculations only for these, in the general case the formulas become ugly. Notice that then  $Q$  and  $P$  depend on time (cf. Heisenberg picture), so to get the right answer for the time-independent Hamiltonian we actually make it depend on time. We discuss this after the calculations.

So now

$$K^t(v(x)) = e^{-i\pi a^{-1}x^2} v(x).$$

Warning: So  $a$  and  $b$  are real numbers i.e. they come without units and we do the calculation without trying to track the units. Even in the calculations physicists do, the tracking of the units appears difficult.

In order to be able to use number theory in our calculations, we assume that in the real world the units are chosen so that  $a$  is an even natural number. We also assume that  $x_0$  and  $x_1$  are non-negative rational numbers  $< a$  such that  $x_0 - x_1$  is an integer (in a sense this assumption is w.l.o.g. since we can make the unit of length as small as we want, see our considerations on the size of our space below).

It is enough to show that

$$\langle x_1 | K^t | x_0 \rangle = (m/2\pi i\hbar t)^{1/2} e^{im(x_0-x_1)^2/2\hbar t}$$

holds in every  $H_N$  for  $N$  divisible enough.

Notice that  $|x_0\rangle$  is  $u(a^{-1}Nx_0)$  and so

$$|x_0\rangle = (1/N)^{1/2} \sum_{n=0}^{N-1} q^{-a^{-1}Nx_0n} v(n)$$

and thus

$$K^t|x_0\rangle = (1/N)^{1/2} \sum_{n=0}^{N-1} q^{-a^{-1}Nx_0n} e^{-i\pi a^{-1}n^2} v(n)$$

i.e.

$$K^t|x_0\rangle = 1/N \sum_{n=0}^{N-1} \sum_{k=0}^{N-1} q^{-a^{-1}Nx_0n} e^{-i\pi a^{-1}n^2} q^{nk} u(k).$$

And so

$$\langle x_1 | K^t | x_0 \rangle = 1/N \sum_{n=0}^{N-1} q^{-a^{-1}Nx_0n} e^{-i\pi a^{-1}n^2} q^{a^{-1}Nx_1n} =$$



$$1/N \sum_{n=0}^{N-1} e^{i\pi(2a^{-1}(x_1-x_0)n-a^{-1}n^2)} =$$

$$1/N \sum_{n=0}^{N-1} e^{i\pi(2(x_1-x_0)n-n^2)/a}.$$

Since  $a$  is even and  $x_0 - x_1$  is an integer,

$$e^{i\pi(a+n)^2/a} = e^{i\pi n^2/a},$$

$$e^{i\pi 2(x_1-x_0)(a+n)/a} = e^{i\pi 2(x_1-x_0)n/a}$$

and also  $N = Na^{-1}a$  and so

$$\langle x_1 | K^t | x_0 \rangle = (1/N) Na^{-1} \sum_{n=0}^{a-1} e^{i\pi(2(x_1-x_0)n-n^2)/a} = a^{-1} \sum_{n=0}^{a-1} e^{i\pi(2(x_1-x_0)n-n^2)/a}.$$

Now we apply number theory: For integers  $c, d, g$ , if  $cg \neq 0$  and  $cg - d$  is even, then

$$\sum_{n=0}^{|g|-1} e^{\pi i(cn^2+dn)/g} = |g/c|^{1/2} e^{\pi i(|cg|-d^2)/4cg} \sum_{n=0}^{|c|-1} e^{-\pi i(gn^2+dn)/c},$$

see [BE]. In our case,  $g = a$ ,  $d = 2(x_1 - x_0)$  and  $c = -1$ . In particular, since  $c = -1$ ,

$$\sum_{n=0}^{|c|-1} e^{-\pi i(gn^2+dn)/c} = 1.$$

So

$$\langle x_1 | K^t | x_0 \rangle = a^{-1} a^{1/2} e^{i\pi(-(1/4)+(x_0-x_1)^2/a)} =$$

$$e^{-i\pi/4} (m/\hbar t)^{1/2} e^{im(x_0-x_1)^2/2\hbar t} =$$

$$(m/2\pi i \hbar t)^{1/2} e^{im(x_0-x_1)^2/2\hbar t}.$$

In the last equation we used the fact that  $(e^{-i\pi/4})^2 = -i = 1/i$ .

Since our calculations depend on making  $P$  time dependent, we study what changes occur when taking time dependence into account. Now

$$H(v(x)) = (P^2/2m)v(x) = ((bx)^2/2m)v(x) = (mx^2/2t^2)v(x)$$

So the Hamiltonian can be written as  $H = t^{-2}H'$  where  $H'$  is time independent. Now the Schrödinger equation demands

$$i\hbar \frac{\partial}{\partial t} K^t = H K^t = t^{-2} H' K^t.$$

Solving this yields

$$K^t = e^{iH'/t\hbar} = e^{iHt/\hbar}.$$

So we define  $K^t$  in  $H_N$  by:

$$K^t(v(x)) = e^{it(bx)^2/2\hbar m}v(x),$$

for all  $0 \leq x < N$ . Using our scaling  $a = \hbar t/m$  and  $b = m/t$  this yields

$$K^t(v(x)) = e^{i\pi a^{-1}x^2}v(x).$$

With calculations like before we get

$$\langle x_1 | K^t | x_0 \rangle = 1/N \sum_{n=0}^{N-1} e^{i\pi(2(x_1-x_0)n+n^2)/a} = a^{-1} \sum_{n=0}^{a-1} e^{i\pi(2(x_1-x_0)n+n^2)/a}.$$

Applying the same quadratic Gauss sum formula, with  $g = a$ ,  $d = 2(x_0 - x_1)$  and  $c = 1$ , we get

$$\begin{aligned} \langle x_1 | K^t | x_0 \rangle &= a^{-1} a^{1/2} e^{\pi i(1/4 - (x_0 - x_1)^2)/a} = \\ &= (m / -2\pi i \hbar t)^{1/2} e^{-im(x_0 - x_1)^2/2\hbar t}, \end{aligned}$$

which is the complex conjugate of the desired result.

### 3. How the model makes reality look

We pause to look at some properties of our model. Clearly, the size of the space (in which the particle lives) depends on time and if  $t = 0$ , the space consists of one point only, see below. In addition to time, the size depends on the mass of the particle. So suppose that the particle is an electron. In our space, we have all the places from the  $Q$ -eigenvalue of

$$(u(0) | N \in X) / D / E$$

to  $Q$ -eigenvalue of

$$(u(N-1) | N \in X) / D / E.$$

The first is zero and the latter is  $a$ . Now  $a = \hbar t/m$ , so at time  $t = 1s$  to calculate  $a$  we compute

$$\frac{6.626 \cdot 10^{-34} m^2 kg / s \cdot 1s}{9.109 \cdot 10^{-31} kg} \approx 7.27 \cdot 10^{-4} m^2.$$

As the units of time and weight cancel out, scaling these does not affect the result. However, scaling length matters: If we measure length in  $cm$  then  $a \approx 7.27$ . Thus the length of the space at time  $1s$  is around  $7.27cm$  At the time  $1h$ ,  $a$  is around

26200 and thus the length of the space is around  $262m$ . Notice that if the mass of the particle increases, the speed of this expansion decreases.

However, if we scale length and choose the unit for length to be  $1mm$ , then it affects the value of  $a$ . At time  $1s$ ,  $a$  is around 727 and so the length of our space at time  $1s$  is around  $73cm$  and at time  $1h$  it is around  $2.6km$ .

Also in the ultraproduct there is an infinite number of possible standard values for momentum operator but the values are not dense in the set of non-negative reals. And the difference between the two consecutive values changes with time. So in a sense, the laws of physics change with time in the ultraproduct. However in the metric ultraproduct we have eigenvectors for all reals, this follows from the embedding we construct in the next section. Also by analogy of what happens with the momentum operator we suspect that the space  $[0, a]$  in which the particle lives is in fact not an interval but rather a circle.

#### 4. An attempt at the harmonic oscillator

We now turn our attention to the Feynman propagator for the harmonic oscillator. The Hamiltonian for the system is

$$H = \frac{P^2}{2m} + \frac{1}{2}m\omega^2 Q^2,$$

where  $\omega$  is the angular frequency. Thus the time evolution operator is

$$K^t = e^{-itH/\hbar} = e^{-\frac{it}{\hbar}(\frac{P^2}{2m} + \frac{1}{2}m\omega^2 Q^2)}.$$

To be able to calculate the Feynman propagator  $\langle x_1 | K^t | x_0 \rangle$  using the bases in the space  $H_N$  we need a factorization that separates the  $P$ - and  $Q$ -parts of the time evolution operator. The literature knows various factorizations of the operator, stated for operators satisfying the commutator relation  $[Q, P] = i\hbar$ . A first obstacle is the already mentioned fact that no finite dimensional Hilbert space can satisfy this relation. But we can find a subspace of  $H$  where the commutation relation holds. This justifies an indirect argument where one calculates the value of the propagator in the spaces  $H_N$  using a factorized version of the time evolution operator that only in the ultraproduct is the right operator. However, here arises another obstacle. The subspace of  $H$  that we find will be  $L_2([0, a])$ , the Hilbert space built of (equivalence classes of) square integrable functions  $h : [0, a] \rightarrow \mathbf{C}$ , for a suitable real  $a$ . If we use the ordinary position and momentum operators

$$Qf(x) = xf(x)$$

and

$$Pf(x) = -i\hbar \frac{d}{dx} f(x)$$

the commutator is densely defined and equal to  $i\hbar$ . But the Weyl commutation relation  $e^{itQ}e^{isP} = e^{-i\hbar st}e^{isP}e^{itQ}$  fails. We will elaborate on this in section 4.4.

Although the obstacles described above restrict the correctness in general of our calculations, we describe our method, as it shows how to calculate the propagator if one has a suitable factorization and as it also seems to give the correct answer.

#### 4.1 The factorization

We use a factorization of the time evolution operator from [QA]. This is stated for operators  $A$ ,  $B$  and  $C$  satisfying  $[A, B] = C$ ,  $[A, C] = 2\gamma A$ ,  $[B, C] = -2\gamma B$  (and these are true of the operators we are interested in, if  $[Q, P] = i\hbar$ ), and it gives coefficients  $\alpha, \beta$  satisfying

$$e^{A+B} = e^{\alpha A} e^{\beta B} e^{\alpha A}.$$

Quijas and Aguilar seem to have some typos in their paper (the third term of the third equation in (55) should be positive, and (59) is not what you get by substitution. But their result satisfies the equations given by the method, producing

$$\alpha = \frac{1}{\sqrt{\gamma}} \tan(\sqrt{\gamma}/2)$$

and

$$\beta = \frac{1}{\sqrt{\gamma}} \sin(\sqrt{\gamma}).$$

Now substituting  $A = -\frac{itm\omega^2}{2\hbar}Q^2$ ,  $B = -\frac{it}{2m\hbar}P^2$  and  $C = -\frac{it^2\omega^2}{2\hbar}(QP + PQ)$  satisfies the commutator requirements with  $\gamma = t^2\omega^2$  and gives  $\alpha = \frac{1}{t\omega} \tan(t\omega/2)$  and  $\beta = \frac{1}{t\omega} \sin(t\omega)$  and thus

$$\begin{aligned} K^t &= e^{-itH/\hbar} = e^{-\frac{it}{\hbar}(\frac{P^2}{2m} + \frac{1}{2}m\omega^2 Q^2)} \\ &= e^{-\frac{im\omega \tan(t\omega/2)}{2\hbar}Q^2} e^{-\frac{i \sin(t\omega)}{2\omega m\hbar}P^2} e^{-\frac{im\omega \tan(t\omega/2)}{2\hbar}Q^2}. \end{aligned}$$

#### 4.2 Finding $L_2([0, a])$ in $H$

As in the first section we pick two positive reals  $a$  and  $b = \hbar/a$  for scaling and only later decide their values. Let  $L_2$  be the usual Hilbert space built of square integrable functions  $f : [0, a] \rightarrow \mathbf{C}$  (as usual, we make no distinction between the function and it's equivalence class) and  $C^\infty \subseteq L_2$  be the vector subspace consisting of those functions that are infinitely differentiable. For integers  $n$ , let  $f_n = a^{-1/2}e^{i2\pi nx/a} \in C^\infty$  and recall that these span  $L_2$ .

We build an embedding of  $L_2$  into  $H$  by first building suitable embeddings into the spaces  $H_N$ . By  $F_N$  we denote the function from  $C^\infty$  to  $H_N$  for which

$$F_N(f) = \sum_{k=0}^{N-1} (a/N)^{1/2} f(a_k) u(k),$$

where  $a_k = ak/N$ . By  $F$  we denote the function from  $C^\infty$  to  $H$  for which  $F(f) = (F_N(f) | N \in \mathbf{N})/D/E$ . It is easy to see that  $F$  is a  $(L_2)$ -norm preserving embedding of the vector space  $C^\infty$  into  $H$ . Thus it can be continued to the whole  $L_2$  and this continuation we also call  $F$ . Clearly  $F$  is an isometric isomorphism between  $L_2$  and some complete subspace of  $H$  (in particular, it preserves the inner product). In addition, an easy calculation shows that  $F_N(f_n) = v(n)$  if  $n \geq 0$

and otherwise  $F_N(f_n) = v(N+n)$  (recall the convention that claims like these need to make sense only for divisible enough  $N$ , in particular, large enough  $N$ ). Thus  $F(f_n) = (v(n) | N \in X)/D/E$  if  $n \geq 0$  and otherwise  $F(f_n) = (v(N+n) | N \in X)/D/E$ .

Now let  $\alpha$  be a real. By  $Q$  we mean the operator  $Q(f)(x) = xf(x)$  in  $L_2$ . We ask whether we can find operators  $A_N^\alpha$  in  $H_N$ ,  $N \in X$ , such that  $F$  is an embedding of the Hilbert space  $L_2$  with  $e^{i\alpha Q^2}$  into the Hilbert space  $H$  with  $\Pi_{N \in X} A_N^\alpha/D/E$ . Since for all  $f \in C^\infty$ ,

$$e^{i\alpha Q^2}(f(x)) = \sum_{m=0}^{\infty} (i\alpha x^2)^m f(x)/m! = e^{i\alpha x^2} f(x),$$

$A_N^\alpha(u(k)) = e^{i\alpha(ka/N)^2} u(k)$  for all  $k < N$ , are such operators.

Similarly, letting  $P(f)(x) = -i\hbar(df/dx)(x)$ , we ask whether we can find operators  $B_N^\alpha$  in  $H_N$ ,  $N \in X$ , such that  $F$  is an embedding of the Hilbert space  $L_2$  with  $e^{i\alpha P^2}$  into the Hilbert space  $H$  with  $\Pi_{N \in X} B_N^\alpha/D/E$ . Here we need to assume that  $\alpha b^2/\pi$  is a rational number (in the case  $k < 0$ , see below). Since for all integers  $k$ ,

$$e^{i\alpha P^2}(f_k(x)) = \sum_{m=0}^{\infty} (i\alpha((i2\pi k/a)i\hbar)^2)^m f_k(x)/m! = e^{i\alpha(bk)^2} f_k(x),$$

$B_N^\alpha(v(k)) = e^{i\alpha(bk)^2} v(k)$  for all  $k < N$ , are such operators. In the case when  $k < 0$  this follows, since  $e^{i\alpha(b(N+k))^2} = e^{i\alpha(bk)^2}$ , when  $N$  is divisible enough (and  $\alpha b^2/\pi$  is a rational number).

### 4.3 Calculating the propagator

According to the factorization from [QA]

$$K^t = e^{i\alpha Q^2} e^{i\beta P^2} e^{i\alpha Q^2},$$

where  $\alpha = -\frac{m\omega \tan(t\omega/2)}{2\hbar}$  and  $\beta = -\frac{\sin(t\omega)}{2\omega m\hbar}$ .

So in the spaces  $H_N$  we use the operators

$$K_N^t = A_N^\alpha B_N^\beta A_N^\alpha$$

which in  $H$  give the operator to which  $e^{i\alpha Q^2} e^{i\beta P^2} e^{i\alpha Q^2}$  is mapped by  $F$  (and thus, if the factorization holds, the operator corresponding to  $K^t$ ).

Now by switching between the bases  $\{u(x) : x < N\}$  and  $\{v(x) : x < N\}$  we can calculate

$$K_N^t(u(x)) = e^{i\alpha(xa/N)^2} \frac{1}{N} \sum_{y=0}^{N-1} \sum_{z=0}^{N-1} q^{y(z-x)} e^{i\beta(by)^2} e^{i\alpha(za/N)^2} u(z).$$

Now remembering  $|x_k\rangle = u(a^{-1}Nx_k)$  we can calculate

$$\langle x_1|K_N^t|x_0\rangle = e^{i\alpha x_0^2} \frac{1}{N} \sum_{y=0}^{N-1} q^{ya^{-1}N(x_1-x_0)} e^{i\beta(by)^2} e^{i\alpha x_1^2}.$$

Now recall that  $b = h/a$  and  $h = 2\pi\hbar$ , so we have

$$\langle x_1|K_N^t|x_0\rangle = e^{i\alpha(x_0^2+x_1^2)} \frac{1}{N} \sum_{y=0}^{N-1} e^{i\pi(2(x_1-x_0)ya^{-1}+\beta\pi^{-1}h^2y^2a^{-2})}.$$

Now we scale by  $a = \frac{h \sin(\omega t)}{m\omega}$  and note that  $\beta/a = -\pi h^{-2}$ , so

$$\langle x_1|K_N^t|x_0\rangle = e^{i\alpha(x_0^2+x_1^2)} \frac{1}{N} \sum_{y=0}^{N-1} e^{i\pi(2(x_1-x_0)ya^{-1}-y^2a^{-1})}.$$

Now as for the free particle case we note if  $a$  is an even integer and  $x_1 - x_0$  is an integer

$$\begin{aligned} e^{i\pi 2(x_1-x_0)(a+y)/a} &= e^{i\pi 2(x_1-x_0)y/a}, \\ e^{i\pi(a+y)^2/a} &= e^{i\pi y^2/a} \end{aligned}$$

so we may write

$$\frac{1}{N} \sum_{y=0}^{N-1} e^{i\pi(2(x_1-x_0)ya^{-1}-y^2a^{-1})} = \frac{1}{N} \frac{N}{a} \sum_{y=0}^{a-1} e^{i\pi(2(x_1-x_0)ya^{-1}-y^2a^{-1})}$$

and use the same Gauss sum formula as before with  $c = -1$ ,  $d = 2(x_1 - x_0)$  and  $g = a$ , rendering

$$\begin{aligned} \langle x_1|K_N^t|x_0\rangle &= e^{i\alpha(x_0^2+x_1^2)} \frac{1}{a} \sum_{y=0}^{a-1} e^{i\pi(2(x_1-x_0)y-y^2)/a} \\ &= e^{i\alpha(x_0^2+x_1^2)} a^{-1} \sqrt{a} e^{-\pi i(a-4(x_1-x_0)^2)/4a} \\ &= \sqrt{a^{-1}} e^{-\pi i/4} e^{i\alpha(x_0^2+x_1^2)+\pi i(x_1-x_0)^2 a^{-1}}, \\ &= \sqrt{\frac{m\omega}{2\pi i\hbar \sin(\omega t)}} e^{\frac{m\omega(\cos(t\omega)(x_0^2+x_1^2)-2x_0x_1)}{2i\hbar \sin(\omega t)}} \end{aligned}$$

which is exactly what the physicists calculate it to be. Above we have used  $\tan(x/2) = (1 - \cos(x))/\sin(x)$ .

#### 4.4 Failure of the Weyl commutation relation

Defining  $Q$  and  $P$  in the standard way in  $L_2([0, a])$  gives the commutation relation  $[Q, P] = i\hbar$ . However, to make  $P$  self adjoint in this space we need to restrict it to the space of continuously differentiable functions  $f$  on  $[0, a]$  with the boundary condition  $f(0) = f(a)$ . Defining  $e^{itP}$  then gives an operator consisting of 'translation with wraparound' (see Example 14.5 of [Ha]), i.e.

$$e^{itP}f(x) = f(x + t\hbar - am_{x,t}).$$

where  $m_{x,t}$  is the unique integer such that

$$0 \leq x + t\hbar - am_{x,t} < a.$$

This is shown by calculating  $e^{itP}$  at its eigenvectors  $f_n$  and extending the operator one gets unitarily. The translation operator does not in general agree with what one gets if one defines the exponential operator via the series

$$e^{itP} = \sum_{n=0}^{\infty} \frac{(itP)^n}{n!},$$

although the definition agrees on the basic vectors  $f_n$ . As the series expression seems to be the key behind the method of [QA], the factorization we use does not hold in general in the space where we use it.

Now we see that the Weyl commutator relation fails:

$$e^{isQ}e^{itP}f(x) = e^{isx}f(x + t\hbar - am_{x,t})$$

and

$$e^{itP}e^{isQ}f(x) = e^{is(x+t\hbar-am_{x,t})}f(x + t\hbar - am_{x,t})$$

so the commutator is  $e^{ist\hbar - isam_{x,t}}$  which is correct only when  $m_{x,t} = 0$ .

But the failure of the Weyl commutator we know already by the Stone-von Neumann theorem. As the Weyl commutator relation for operators  $Q$  and  $P$  on a separable space imply these must (essentially) be the standard  $Q$  and  $P$  on  $L_2(\mathbf{R})$ , we know we cannot satisfy the relation. On  $L_2(\mathbf{R})$  both  $Q$  and  $P$  are unbounded and without eigenvectors, whereas on  $L_2([0, a])$ ,  $Q$  is bounded and  $P$  has eigenvectors.

The problem with exponential operators reoccurs with  $e^{itP^2}$ , that again consists of shifting, but this time shifts different eigenvectors with different amounts.

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